MOUNT LAWLEY SENIOR HIGH SCHOOL

Semester 2 Examination, 2011

Question/Answer Booklet

MATHEMATICS SPECIALIST
MAS 3C/3D

Section Two
Calculator- assumed

NAME Key

Time allowed for this section
Reading time before commencing work: 10 minutes
Working time for paper: 100 minutes

Material required/recommended for this section
To be provided by the supervisor
This Question/Answer booklet
Formula sheet (retained from Section One)

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the Curriculum Council for this course.

Important note to candidates

No other items may be used in this section of the examination. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

This section has **thirteen (13)** questions.
Answer **all** questions. Write your answers in the spaces provided.

Working time for this section is 100 minutes.

Question 7 **(5 marks)**

The temperature, I°C, of a liquid in an insulated flask at any time t seconds can be described by the differential equation $\frac{dI}{dt} = -0.003I$.

(a) How long will it take for the liquid in the flask to fall by 10%? [2]

$I = I_0 e^{-0.003t}$

need t when $I = 0.9 I_0$

ie $e^{-0.003t} = 0.9$

Calculator $\Rightarrow$ $t = 35.12$ secs ✓✓

$\approx$ Just over 35 seconds

The temperature of a liquid in another, uninsulated, flask is decreasing exponentially at a continuous percentage decay rate of 0.75%. Given the initial temperatures of the liquids in the insulated and uninsulated flasks are 65°C and 95°C respectively and knowing that in the cooling process there will be multiple times when the liquids will be at the same temperature,

(b) determine the first two times when the difference in temperature between the two liquids is 10°C. [3]

insulated : $I_{in} = 65 e^{-0.003t}$

uninsulated: $I_u = 95 e^{-0.0075t}$

Need t when $|65e^{-0.003t} - 95e^{-0.0075t}| = 10$ ✓

Calculator: $t = 47.98$ ✓, 144.55 ✓ secs.

Question 8

(6 marks)

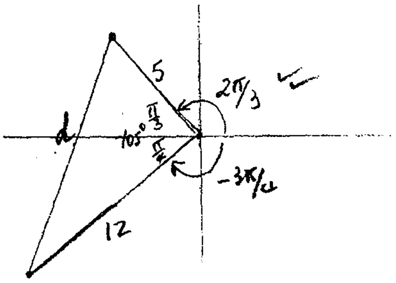
(a) Find the distance between the points with polar coordinates $\left(5, \frac{2\pi}{3}\right)$ and $\left(12, -\frac{3\pi}{4}\right)$, where distances are in centimetres and angles in radians. [3]

Either $d = \sqrt{5^2 + 12^2 - 2 \times 5 \times 12 \cos 105^\circ}$

$= 14.14$ (2dp)

OR. Geometry mode

$d = 14.14$ ✓

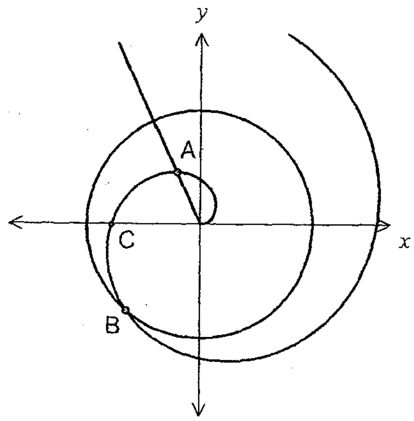


(b) The graphs of

$\theta = \alpha$, $r = \beta$ and $r = n\theta$

are shown together with the points A and B which have polar coordinates of $(1, 2)$ and $(\beta, 4)$ respectively.

Find the values of β, n and the polar coordinates of point C. [3]



Spiral $r = n\theta$ passes thru $A(1, 2)$

ie $1 = n \times 2 \Rightarrow n = \frac{1}{2}$ ✓

$\therefore r = \frac{\theta}{2}$ and $B(\beta, 4)$ is on the spiral

$\therefore \beta = \frac{4}{2} \Rightarrow \beta = 2$ ✓

C is also on the and $\theta = \pi \Rightarrow r = \frac{\pi}{2}$

$\therefore$ C is $\left(\frac{\pi}{2}, \pi\right)$ ✓

Question 9 **(8 marks)**

The point A has position vector $3\mathbf{i}-2\mathbf{j}+4\mathbf{k}$ and O is at the origin.

(a) Find the value of a if the vectors $\overrightarrow{OA}$ and $a\mathbf{i}+3\mathbf{j}-3\mathbf{k}$ are perpendicular. [1]

$$\therefore \begin{bmatrix}3\\-2\\4\end{bmatrix} \bullet \begin{bmatrix}a\\3\\-3\end{bmatrix} = 0$$

ie $3a-6-12=0$

$\therefore \underline{a=6}$ ✓

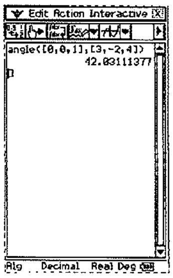
(b) Find the size of the angle between $\overrightarrow{OA}$ and the z-axis, to the nearest degree. [2]

$\begin{bmatrix}0\\0\\1\end{bmatrix}$ is on the z-axis

$$\therefore \cos\theta = \frac{\begin{bmatrix}3\\-2\\4\end{bmatrix} \cdot \begin{bmatrix}0\\0\\1\end{bmatrix}}{\sqrt{29} \times 1}$$

ie $\cos\theta = \frac{4}{\sqrt{29}}$

$\therefore \theta \doteq 42^\circ$



(c) Find the value of b if the points $(7, b, 2)$ and $(-1, 2, 5)$ are in the plane perpendicular to the vector $\overrightarrow{OA}$. [2]

$$\begin{bmatrix}7\\b\\2\end{bmatrix} - \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}8\\b-2\\-3\end{bmatrix}$$ is in the plane $\perp$ to OA.

$$\therefore \begin{bmatrix}3\\-2\\4\end{bmatrix} \cdot \begin{bmatrix}8\\b-2\\-3\end{bmatrix} = 0$$

ie $24-2(b-2)-12=0$

ie $\underline{b=8}$ ✓✓

(d) Find the value of c if the point $(15, -14, c)$ lies on the straight line through A and the point $(-1, 2, 5)$. [3]

Vector in direction of line is $\begin{bmatrix}3\\-2\\4\end{bmatrix} - \begin{bmatrix}-1\\2\\5\end{bmatrix} = \begin{bmatrix}4\\-4\\-1\end{bmatrix}$

$\therefore$ Eq^n of line is $\underset{\sim}{r} = \begin{bmatrix}3\\-2\\4\end{bmatrix} + \lambda \begin{bmatrix}4\\-4\\-1\end{bmatrix}$ ✓

ie $\underset{\sim}{r} = \begin{bmatrix}3+4\lambda\\-2-4\lambda\\4-\lambda\end{bmatrix}$

$\therefore \begin{bmatrix}15\\-14\\c\end{bmatrix} = \begin{bmatrix}3+4\lambda\\-2-4\lambda\\4-\lambda\end{bmatrix}$ ✓ $\Rightarrow$ $3+4\lambda=15$

$\therefore \lambda = 3$

and $c = 4-\lambda$

$\therefore \underline{c=1}$ ✓

Question 10 **(5 marks)**

When an object is at a distance u cm from a lens of focal length 20 cm, an image is created at a distance of v cm from the lens.

The variables are related by the formula $\frac{1}{20}=\frac{1}{u}+\frac{1}{v}$.

An object is moving with a constant speed of 2 cm/s towards the lens.

At the instant when the image is 30 cm from the lens, in what direction and with what speed is ~~it~~ the image moving?

$$\frac{1}{20} = \frac{1}{u} + \frac{1}{v}$$

Need $\frac{dv}{dt}$ when $v = 30$.

given $\frac{du}{dt} = -2$

diff. w.r.t. t

ie $0 = -\frac{1}{u^2} \cdot \frac{du}{dt} - \frac{1}{v^2} \cdot \frac{dv}{dt}$

When $v = 30$

$\frac{1}{20} = \frac{1}{u} + \frac{1}{30}$

$\Rightarrow u = 60$

$\therefore 0 = -\frac{1}{60^2} \cdot (-2) - \frac{1}{30^2} \cdot \frac{dv}{dt}$

$\therefore \frac{1}{30^2} \cdot \frac{dv}{dt} = \frac{2}{60^2}$

ie $\frac{dv}{dt} = \frac{2 \times 30 \times 30}{60 \times 60}$

$\frac{dv}{dt} = \frac{1}{2}$ cm/sec away from the lens.

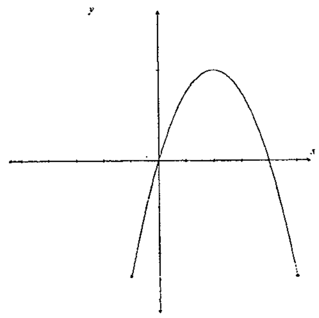
OR $\frac{1}{20} = \frac{1}{u} + \frac{1}{v} \Rightarrow u = \frac{20v}{(v-20)} \Rightarrow \frac{du}{dv} = -\frac{400}{(v-20)^2}$

Now $\frac{dv}{dt} = \frac{dv}{du} \times \frac{du}{dt} \Rightarrow \frac{dv}{dt} = -\frac{(v-20)^2}{400} \times -2$

$v = 30 \Rightarrow \frac{dv}{dt} = \frac{1}{2}$

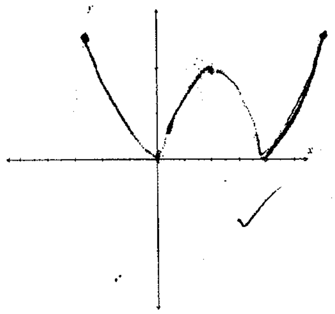
Question 11 (~~5~~ 4 marks)

(a) The graph of $y = f(x)$ is shown.



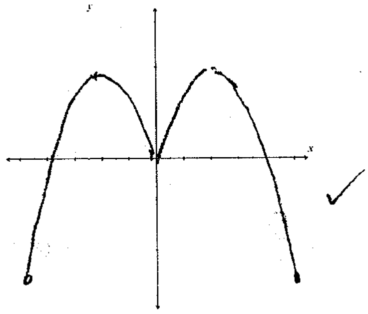
Sketch the graphs of [2]

(i) $y = |f(x)|$



✓

(ii) $y = f(|x|)$



✓

(b) The equation $|ax+b| = |x-4|$ has solutions $x = -0.2$ and $x = -3$.

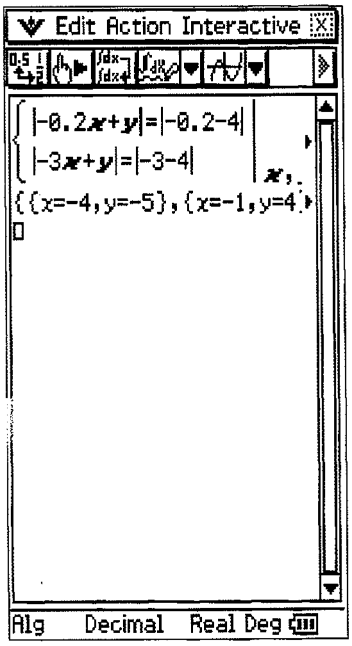
Find all the simultaneous values values of a and b. [~~3~~] 2

$x = -0.2 \Rightarrow |-0.2a + b| = |-0.2-4|$

$x = -3 \Rightarrow |-3a + b| = |-3-4|$

Calculator:

a	b
4	-5
-1	4
1	-4
-4	5



Question 12

(~~4~~ 5 marks)

Prove that $\dfrac{1+sin(2\theta)-cos(2\theta)}{1+sin(2\theta)+cos(2\theta)}=tan\theta$.

$$LHS = \frac{1 + 2\sin\theta\cos\theta - (1 - 2\sin^2\theta)}{1 + 2\sin\theta\cos\theta + (2\cos^2\theta - 1)}$$

$$= \frac{1 + 2\sin\theta\cos\theta - 1 + 2\sin^2\theta}{1 + 2\sin\theta\cos\theta + 2\cos^2\theta - 1}$$

$$= \frac{2\sin\theta\cos\theta + 2\sin^2\theta}{2\sin\theta\cos\theta + 2\cos^2\theta}$$

$$= \frac{\cancel{2}\sin\theta\,\cancel{(\cos\theta + \sin\theta)}}{\cancel{2}\cos\theta\,\cancel{(\sin\theta + \cos\theta)}}$$

$$= \frac{\sin\theta}{\cos\theta}$$

$$= \underline{\tan\theta}$$

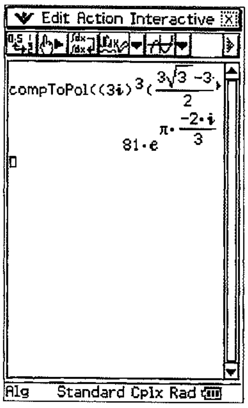
Question 13 **(6 marks)**

Two complex numbers are given by $u = 3i$ and $v = \dfrac{3\sqrt{3} - 3i}{2}$.

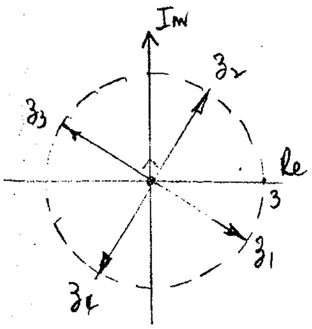
(a) Express $u^3 v$ in the form $r\,cis\theta$. [2]

Calculator:

$81\ cis\left(-\frac{2\pi}{3}\right)$



(b) List all solutions for z in exponential form, given that $z^4 = u^3 v$. [2]



$z^4 = 81\ cis\left(-\frac{2\pi}{3}\right)$

$z^4 = 81\ e^{i\left(-\frac{2\pi}{3}\right)}$

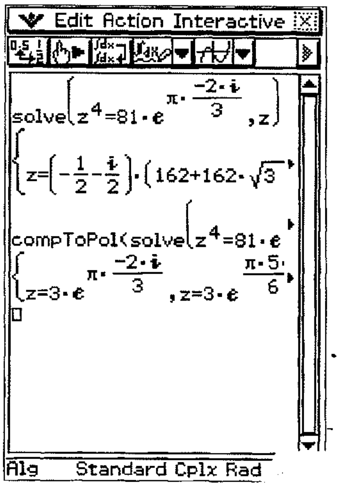
$\therefore z_1 = 3\ e^{i\left(-\frac{\pi}{6}\right)}$

$z_2 = 3\ e^{i\left(\frac{\pi}{3}\right)}$

$z_3 = 3\ e^{i\left(\frac{5\pi}{6}\right)}$

$z_4 = 3\ e^{i\left(-\frac{2\pi}{3}\right)}$

✓✓



(c) Show that the sum of all the solutions from part (b) is 0. [2]

$z_1 + z_3 = 0$. same magnitude but opposite directions.

$z_2 + z_4 = 0$ 〃 ✓✓

Question 14 **(5 marks)**

At a school with 108 boarders, boarders can either eat breakfast or not eat breakfast.

The canteen manager estimates that of those boarders who eat breakfast one morning, 5% of them will not eat breakfast the next morning and of those boarders who do not eat breakfast one morning, 55% of them eat breakfast the following morning.

(a) If 55 boarders eat breakfast on Monday, how many boarders should the canteen manager expect to eat breakfast on Wednesday? [3]

From

$$\text{To}\quad \begin{array}{cc} & \begin{array}{cc}\text{Brekky Today} & \text{No brekky today}\end{array} \\ \begin{array}{c}\text{Brekky Tomorrow}\\ \text{No Brekky Tomorrow}\end{array} & \begin{bmatrix} 0.95 & 0.55 \\ 0.05 & 0.45 \end{bmatrix} \end{array} = T \checkmark$$

$$\text{Tues} \Rightarrow T\begin{bmatrix}55\\53\end{bmatrix}$$

$$\text{Wed} \Rightarrow T^2\begin{bmatrix}55\\53\end{bmatrix} = \begin{bmatrix}91.96\\16.04\end{bmatrix} \checkmark$$

$\therefore$ Expect 92 for Brekky on Wednesday ✓

(b) In the long term, what proportion of boarders can be expected to eat breakfast? [2]

$$T^n\begin{bmatrix}55\\53\end{bmatrix} = \begin{bmatrix}99\\9\end{bmatrix} \text{ for } n \text{ large}$$

$\therefore$ proportion is $\dfrac{99}{99+9} = \dfrac{11}{12}$. ✓✓

OR

$$\frac{0.55}{0.55+0.05} = \frac{11}{12} \quad \checkmark\checkmark$$

$91\frac{2}{3}\%$ OK.

OR

$$\begin{bmatrix}0.95 & 0.55\\0.05 & 0.45\end{bmatrix}^{\text{Large Power (Say 30)}} = \begin{bmatrix}0.9166\ldots & 0.9166\ldots\\0.0833\ldots & 0.0833\ldots\end{bmatrix}$$

Question 15 **(~~6~~ 7 marks)**

The graphs of the function $f(x) = 2\ln(x+k)$, where k is a constant, and its inverse $y = f^{-1}(x)$, intersect where $x = -2$ and at one other point.

(a) Given the graphs of $y = f(x)$ and the inverse $y = f^{-1}(x)$ are symmetrical about the line $y = x$, show that the exact value of k is $\frac{1}{e} + 2$. [2]

$x = -2 \Rightarrow$ $\therefore y = -2$.

$2\ln(k-2) = -2$

ie $\ln(k-2) = -1$

$\therefore k - 2 = e^{-1}$

$k = \frac{1}{e} + 2$

calculator ok

(b) Sketch the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$ on the axes below, showing any asymptotes and showing the coordinates of all points of intersection and axes-intercepts. Where appropriate give values correct to 2 decimal places. ~~[4]~~ 5

$f: \; y = 2\ln(x+k)$

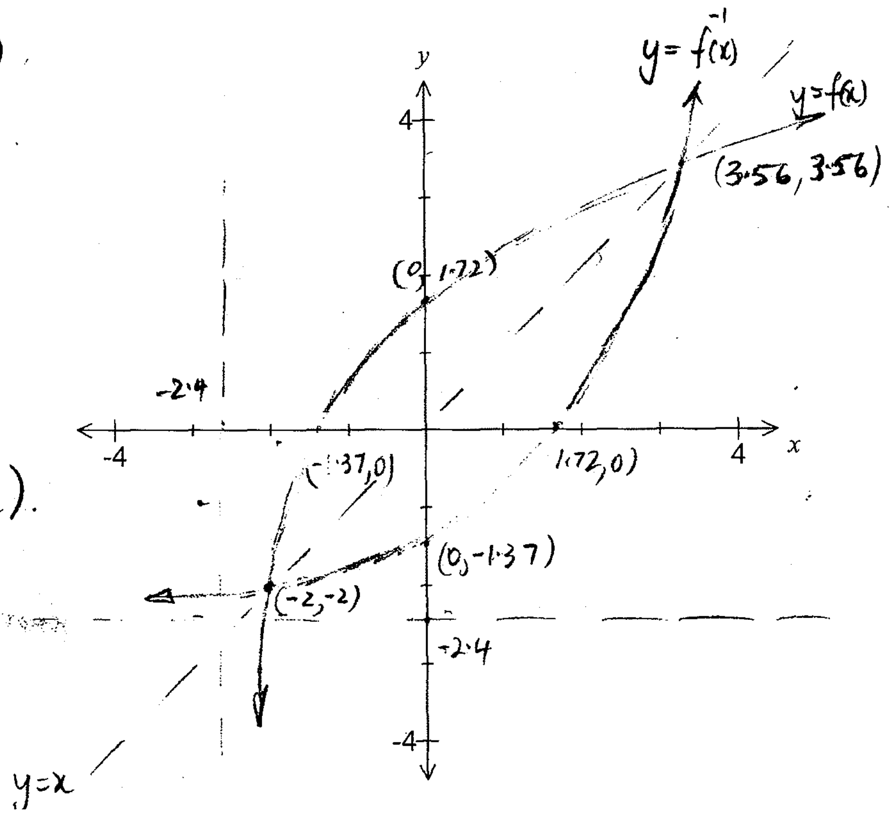
$f^{-1}: \; x = 2\ln(y+k)$

$\therefore \frac{x}{2} = \ln(y+k)$

$\therefore y + k = e^{x/2}$

$\therefore y = e^{x/2} - k$

ie $f^{-1}(x) = e^{x/2} - (2 + \frac{1}{e})$.



Question 16 **(5 marks)**

(a) The matrix equation $AX = B$ could be used to solve the following system of equations.

$$2a+3b=c-5$$
$$b-2c-4a=1$$
$$5=c+b$$

Write down suitable matrices for A, X and B then illustrate how to solve the system of equations using a matrix method. **(Do not solve the equations)** [2]

$2a+3b-c=-5$
$-4a+b-2c=1$
$b+c=5$

$$\Rightarrow \begin{bmatrix} 2 & -3 & -1 \\ -4 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} \checkmark$$

$A \times X = B$

$$\therefore \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2 & -3 & -1 \\ -4 & 1 & -2 \\ 0 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -5 \\ 1 \\ 5 \end{bmatrix} \checkmark$$

(b) If $PQ = 3P + I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $Q = \begin{bmatrix} 7 & -9 \\ 5 & -8 \end{bmatrix}$, find matrix P. [3]

Now $PQ - 3P = I$

ie $P(Q-3I) = I$

$\therefore P = I(Q-3I)^{-1}$

ie $P = (Q-3I)^{-1}$ ✓✓

Calculator $\therefore P = \begin{bmatrix} -11 & 9 \\ -5 & 4 \end{bmatrix}$ ✓

Question 17 **(7 marks)**

The displacement $x(t)$ metres, of a small particle undergoing simple harmonic motion is given by $x(t) = A\cos(\omega t) + B\sin(\omega t)$, where A, B and ω are positive constants.

(a) Show that $x''(t) + \omega^2 x(t) = 0$ [2]

$$x' = -A\omega \sin(\omega t) + B\omega \cos(\omega t)$$

$$x'' = -A\omega^2 \cos(\omega t) - B\omega^2 \sin(\omega t).$$

ie $x'' = -\omega^2 [A\cos(\omega t) + B\sin(\omega t)]$

ie $x'' = -\omega^2 x(t).$

$\therefore x'' + \omega^2 x(t) = 0.$

The particle completes 2.5 cycles every second and initially has a displacement of 1.5 m and a velocity of 7.5 ms^{-1}.

(b) Determine the exact values of the constants A, B and ω. [3]

2.5 cps $\Rightarrow$ period of 0.4 secs.

$\therefore \frac{2\pi}{\omega} = 0.4 \Rightarrow \omega = 5\pi$ ✓

$t=0, x=1.5 \Rightarrow 1.5 = A.\cos 0 + B\sin 0.$

$\therefore \underline{A = 1.5}$ ✓

$t=0, v=7.5 \Rightarrow 7.5 = -1.5\times 5\pi \sin 0 + B.\cos 0.$

$\therefore 7.5 = B \times 5\pi$

$\therefore B = \frac{7.5}{5\pi} \Rightarrow \frac{3}{2\pi}.$ ✓

(c) What is the amplitude of motion, correct to the nearest millimetre? [2]

✓

Amplitude $= \sqrt{1.5^2 + \left(\frac{3}{2\pi}\right)^2}$

≈ 1.574 m to nearest mm.

✓

Question 18 **(9 marks)**

Relative to itself, an anti-ballistic missile (ABM) launch site detects a ballistic missile at $11\mathbf{i}-18\mathbf{j}+10\mathbf{k}$ km headed at constant velocity for a target at $35\mathbf{i}+14\mathbf{j}+\mathbf{k}$ km.

The ballistic missile is expected to hit the target in 50 seconds.

(a) How close does the ballistic missile come to the ABM launch site? [5]

Direction of ballistic missile and displacement in 50 secs $\begin{bmatrix}35\\14\\1\end{bmatrix} - \begin{bmatrix}11\\-18\\10\end{bmatrix} = \begin{bmatrix}24\\32\\-9\end{bmatrix}$ km

$\therefore$ Velocity of ballistic missile is $\frac{1}{50}\begin{bmatrix}24\\32\\-9\end{bmatrix}$ km/sec $\left(\underset{\sim}{v} = \frac{\text{displ}}{\text{time}}\right)$

$\therefore$ Position of ballistic missile "$\underset{\sim}{r}$" relative to launch site is $\underset{\sim}{r} = \begin{bmatrix}11\\-18\\10\end{bmatrix} + \frac{t}{50}\begin{bmatrix}24\\32\\-9\end{bmatrix}$ "t" seconds later.

ie $\underset{\sim}{r} = \begin{bmatrix}11 + 0.48t\\-18 + 0.64t\\10 - 0.18t\end{bmatrix}$

with $\underset{\sim}{r}$ being the "separation vector" between the launch site and ballistic missile.

Need $|\underset{\sim}{r}|$ min where

$$|\underset{\sim}{r}| = \sqrt{(11+0.48t)^2 + (-18+0.64t)^2 + (10-0.18t)^2}$$

Calculator: "F Min"

$\Rightarrow$ Min distance $\underline{21.186\text{ km}}$ after 11.96 secs

Question 18 (continued)

The launch site plans to fire an ABM to hit the ballistic missile. The hit is timed to take place at the instant the ballistic missile comes within 8km of the target.

(b) Assuming the ABM instantly achieves a constant velocity of 1150 ms^{-1} as it is launched, how long from the time of detection should the defence site fire it? Give your answer to the nearest tenth of a second. [4]

$|24i + 32j - 9k| = 41$ km $\Rightarrow$ distance between first detection and target.

Will collide with ABM 8 km from target.

$\therefore$ 33 km from first detection and after

$\frac{33}{41} \times 50 \doteq 40.244$ seconds

$\therefore$ distance travelled from detection will be

$$\left| \begin{bmatrix} 11 \\ -18 \\ 10 \end{bmatrix} + \frac{33}{41} \begin{bmatrix} 24 \\ 32 \\ -9 \end{bmatrix} \right| = \left| \begin{matrix} 30.317 \\ 7.756 \\ 2.756 \end{matrix} \right|$$

≈ 31.415 km

$\therefore$ Time for ABM to hit missile $= \frac{31.415}{1.15}$

1150 ms^{-1} $= 1.15$ km/sec

$= 27.317$ secs

$\therefore$ Must launch $40.244 - 27.317 = 12.93$ secs from the time of detection

Question ~~18~~ 19 **(8 marks)**

Every odd integer I can be written as ten times some integer, n, plus some constant c. That is, $I = 10n + c$, where $c = 1, 3, 5, 7, 9$.

(a) Show how the odd integers 237 and -35 can be written this way. [1]

$237 = 10 \times 23 + 7$

$-35 = 10 \times (-4) + 5$ ✓

(b) By exhausting the five different cases for c prove that the square of every odd integer ends in 1, 5 or 9. [7]

$I^2 = (10n + c)^2$

$c = 1$, $I^2 = (10n+1)^2$
$= 100n^2 + 20n + 1$
$= 10(10n^2 + 2n) + 1$ must end in 1

$c = 3$, $I^2 = (10n+3)^2$
$= 100n^2 + 60n + 9$
$= 10(10n^2 + 6n) + 9$ must end in 9

$c = 5$, $I^2 = (10n+5)^2$
$= 100n^2 + 100n + 25$
$= 100(n^2 + n) + 25$, must end in 5.

$c = 7$. $I^2 = (10n+7)^2$
$= 100n^2 + 140n + 49$
$= 10(10n^2 + 14n) + 49$, must end in 9.

$c = 9$. $I^2 = (10n+9)^2$
$= 100n^2 + 180n + 81$
$= 10(10n^2 + 18n) + 81$ must end in 1

$\therefore$ square of every odd integer must end in 1, 5 or 9.